octospores; but he failed to interpret their true significance as reproductive organs, and laid down his pen under the firm conviction that the cystocarpic fruit was entirely absent in Porphyra. Thuret's representation of this kind of fruit proved that Janczewski was mistaken. Dr. Berthold mentions that he was fortunate enough to obtain by his researches at the Zoological Station at Naples satisfactory proof that the reproductive processes in the Bangiaceæ correspond exactly with those of the other Florideæ; he further states (p. 21) that they are true Florideæ, but that they undoubtedly occupy the very lowest position in the class

The first part of the work describes at some length the structure of the vegetative thallus of each of the three genera. A minute description follows of the organs of fructification, namely, the tetraspores, cystocarps, and antheridia, and of the mode in which the cystocarps are fertilised. The fructification of all the genera is illustrated by a plate containing twenty-five figures. We have then an account of the germination of the spores and of their development into plants, followed by observations on the systematic position occupied by the Bangiaceæ and their relation to the Chlorosperms. To these are added descriptions of the genera and species, with a notice of the habitat and time of appearance of the several species.

This very interesting work concludes with some remarks on Goniotrichum, and short descriptions of the two species *G. elegans* (*Bangia elegans* of the "Phyc. Brit.," Pl. ccxlvi.) and *G. dichotomum*. MARY P. MERRIFIELD

## GAUSS AND THE LATE PROFESSOR SMITH

IN the centenary notice of Gauss (NATURE, vol. xv. pp. 533-537) I more than once refer to notes placed in my hands by the late Prof. Henry Smith. These took the form of two MSS. (*A*), (*B*). The former of these I used in its entirety (p. 537), the latter I withheld, with Prof. Smith's sanction, on account of the length to which the article had already extended. Many mathematicians may now like to read these further criticisms on Gauss by such a kindred genius. R. TUCKER

We proceed to give brief references to some of the most important points which have caused a new epoch in certain branches of analysis to date from the publication of the "Disquisitiones Arithmeticæ," and from the researches with which, some years later, Gauss supplemented or further developed the theories contained in that work. It may be proper to premise that Gauss found the theory of numbers as Euler and Lagrange had left it. Of these the former had enriched it with a multitude of results, relating to diophantine problems, to the theory of the residues of powers, and to binary quadratic forms; the latter had given the character of a general theory to some at least of these results, by his discovery of the reduction of quadratic forms, and of the true principles of the solution of indeterminate equations of the second degree. Legendre (with many additions of his own) had endeavoured to arrange as much as possible of these scattered fragments of the science into a systematic whole in his "Essai sur La Théorie des Nombres." But the "Disquisitiones Arithmeticæ" was in the press when this important treatise appeared, and what in it was new to others was already known to Gauss.

The first section of the "Disquisitiones," "De Numerorum Augmentiâ in genere," occupies hardly more than four pages of the quarto edition, and is of the most elementary character. Nevertheless, the definition and the elementary properties of a congruence, which were for the first time given in it, have exercised an immense influence over all the branches of the higher arithmetic; an influence which is perhaps surprising when we remember that it is a question of notation only, and that (as Gauss has said himself in a letter to Schumacher) nothing can be done with this notation which cannot (though less conveniently) be done without it.

The second section, "De Congruentiis Primi Gradus," contains applications of the definition and of the elementary properties of congruences to linear congruences, and to systems of such congruences. The problems solved in it are of an elementary kind, and may be regarded as either well known, or as lying within the scope of what was well known, at the time of the publication of the "Disquisitiones Arithmeticæ."

The same remark applies to the third section, "De Residuis Potestatum," which, notwithstanding the immense advantage of clearness and simplicity obtained by the use of the congruential notation, may be said to lie almost wholly within the aid of ideas to be found in Euler's memoirs. The demonstration of the existence of primitive roots (a demonstration which Euler had failed in rendering rigorous), is, however, a very noticeable exception.

The fourth section—"De Congruentiis Secundi Gradus"—opens with an exposition of the elementary theorems relating to quadratic residues and non-residues; and so far we are still entirely within the ground already occupied by Euler. But the greater part of this section is occupied with a research which of itself alone would have placed Gauss in the first rank of mathematicians. "If $p$ and $q$ are positive uneven prime numbers, $p$ has the same quadratic character with regard to $p$ that $q$ has with regard to $p$, except when $p$ and $q$ are both of the form $4n + 3$, in which case the two characters are always opposite instead of identical." This is the celebrated Fundamental Theorem of Gauss, known also as the law of quadratic reciprocity of Legendre. Gauss discovered it (by induction) in March, 1795, before he was eighteen; the proof given of it in this section he discovered in April of the year following. He cannot at the earlier date have been aware that the theorem had been already enunciated (though in a somewhat complex form) by Euler; and that Legendre had attempted, though unsuccessfully, to prove it in the *Mémoires of the Academy of Paris* for 1784. But the question to whom the priority of the enunciation is due is of even less moment than questions of priority usually are; for the discovery of the theorem by induction was easy, whereas any rigorous demonstration of it involved apparently insuperable difficulties. Gauss was not content with vanquishing these difficulties once for all in the fourth section. In the fifth section he returns to it again, and obtains another demonstration reposing on entirely different, but perhaps still less elementary, principles. In January of the year 1808 he submitted a third demonstration to the Royal Society at Göttingen; a fourth in August of the same year; a fifth and sixth in February, 1817. It is no wonder that he should have felt a sort of personal attachment to a theorem which he had made so completely his own, and which he used to call the gem of the higher arithmetic. His six demonstrations remained for some time the only efforts in this direction; but the subject subsequently attracted the attention of other eminent mathematicians, and several proofs, differing substantially from one another and from those of Gauss, have been given by Jacobi and Eisenstein in Germany, and by M. Liouville in France, the simplest of all perhaps being that which has been given by a Russian mathematician, M. Zeller, and which is of the same general character as the third proof of Gauss (see *Messenger of Mathematics*, vol. v. pp. 140-3, 1876). It would certainly be impossible to exaggerate the important influence which this theorem has had on the subsequent development of arithmetic, and the discovery of its demonstration by Gauss must certainly be regarded (indeed it was so regarded by himself) as one of his greatest scientific achievements.

The fifth section—"De Formis Æquationibusque Inde-

terminatis Secundi Gradus"—consists, as has been said with great truth by Dirichlet, of two distinct parts. Of these the first (Arts. 153-222) contains a complete exposition of the theory of binary quadratic forms, as far as it was known from the researches of Euler and Lagrange; although even these known results are completed in many respects and are exhibited from a new and independent point of view. The second part (Arts. 223-305) contains investigations which are entirely Gauss's own: the distribution of the classes of binary forms into genera; the determination of the number of ambiguous classes; the demonstration that only one-half of the genera possible *a priori* actually exist, and the proof of the fundamental theorem deduced from this result; a disquisition on ternary quadratic forms, introduced as a digression; the theory of the decomposition of numbers into three squares; the solution of indeterminate equations of the second degree in rational numbers; the determination of the mean number of the genera and classes; the distinction between regular and irregular determinants. Such is a brief list of the subjects treated of in these marvellous pages, each of which has been the starting-point of long series of important researches by subsequent mathematicians.

In the *Additamenta* to this section, Gauss intimates that he had succeeded in determining the relations between the determinant and the number of classes; and in a manuscript note he characteristically adds: "Ex voto nobis sic successit ut nihil amplius desiderandum supersit, Nov. 30–Dec. 3, 1800." It is remarkable that he should never have published the wonderful researches to which he here alludes. These researches first saw the light sixty-three years later in the second volume of the collected edition of his works; but the theorem to which they refer had in the interval been rediscovered and demonstrated by Lejeune Dirichlet. The demonstration of Dirichlet had been to a certain extent simplified by M. Hermite, and the form of demonstration found in Gauss's papers after his death approaches very nearly to that adopted by M. Hermite.

The sixth section contains some applications of arithmetical principles to various practical questions. Of these the first two are comparatively elementary, and relate to the resolution of fractions into simpler fractions, and to the conversion of vulgar into decimal fractions; the others consist in systematic methods of abbreviating certain tentative processes, such as the solution of quadratic congruences, the decomposition of numbers into their prime factors, the solution of certain indeterminate equations, &c. The methods of Gauss still remain the least unsatisfactory that have been proposed for the indirect treatment of these difficult problems, of which any direct solution seems impossible.

The seventh section, "De Æquationibus Circuli Sectiones Definientibus," is that which at once made the reputation of the "Disquisitiones Arithmeticæ." It is not too much to say that till the time of Jacobi the profound researches of the fourth and fifth sections were passed over with almost universal neglect. But the well-known theory of the division of the circle comprised in this section was received with great and deserved enthusiasm as a memorable addition to the theory of equations and to the geometry of the circle. One of Gauss's manuscript notes is interesting, "Circulum in 17 partes divisibilem esse geometrice, deteximus 1796, Mart. 30," because it shows that he was not yet nineteen when he made this great discovery. Even more remarkable, however, is a passage in the first article of the section (Art. 335), in which Gauss observes that the principles of his method are applicable to many other functions besides the circular functions, and in particular to the transcendents dependent on the integral $\int \frac{dx}{\sqrt{1-x^4}}$. This almost casual remark shows (as Jacobi long since observed) that Gauss, at the date of the publication of the "Disquisitiones Arithmeticæ," had already examined the nature and properties of the elliptic functions (the inverses of the elliptic integrals), and had discovered their fundamental property, that of double periodicity. This observation of Jacobi's is amply confirmed by the papers on elliptic transcendents now published in the third volume of Gauss's collected works.

The "Disquisitiones Arithmeticæ" were to have included an eighth section. This eighth section was at first intended to contain a complete theory of congruences, but subsequently Gauss appears to have proposed to continue the work by a more complete discussion of the theory of the division of the circle. Manuscript drafts on each of these subjects were found among his papers; the first of them is especially interesting, as it treats of the general theory of congruences from a point of view closely allied to that subsequently taken by Evariste Galois and by MM. Serret and Dedekind. This draft appears to belong to the years 1797 and 1798.

To complete this hasty outline of the arithmetical works of Gauss it only remains to mention (1) the remarkable geometrical interpretation of the arithmetical theory of positive binary and ternary quadratic forms, which will be found in his review (1831) of the work of L. Seeber ("Werke," vol. ii. p. 188), and (2) the two important memoirs on the theory of biquadratic residues (1825 and 1831). In the second of these memoirs Gauss introduces into arithmetic *complex* numbers of the form $a + bi$. He finds that in this complex theory every prime number of the form $4n + 1$ is to be regarded as composite, because, being the sum of two squares, *e.g.* $p = a^2 + b^2$, it is a product of two conjugate factors, $p = (a + bi)(a - bi)$. Thus the true primes of the complex theory may be defined to be the real primes of the form $4n + 3$, and the imaginary factors of real primes of the form $4n + 1$. Availing himself of this definition, Gauss discovered a theorem of biquadratic reciprocity between any two prime numbers, no less simple than the quadratic law, viz. "If $p_1$ and $p_2$ are two primary prime numbers, the biquadratic character of $p_1$ with regard to $p_2$ is the same as that of $p_2$ with regard to $p_1$."

Both this theorem of reciprocity itself and the introduction of imaginary integers upon which it depends are memorable in the history of arithmetic for the number and variety of the researches to which they have given rise.

It may perhaps seem remarkable that Gauss should have devoted so few memoirs to subjects of an algebraical character. If we except a comparatively unimportant paper on Descartes' rule of signs which appeared in *Crelle's Journal* in the year 1828, his only algebraical memoirs relate to the theorem that every equation has a root. Of this he gave no less than three distinct demonstrations, one in 1799, one in 1815, and one in 1816; the demonstration of 1799 was given in his first published paper—his dissertation as a candidate for the degree of Doctor of Philosophy in the University of Göttingen. This demonstration he repeated over again in 1849, with certain changes and simplification. There can be no question that these three demonstrations are prior to any other, though for various reasons those subsequently given by Cauchy have been justly preferred for the purpose of insertion in our modern text-books.

## ANTHROPOLOGY IN AMERICA

WE cannot speak very highly of Prof. Otis T. Mason's "Account of Progress in Anthropology in the Year 1881," which was originally embodied in the Smithsonian Report for that year, and is now issued in a separate form. There is no comprehensive survey of the work done in this wide field during the period indicated, and the bibliography, of which the paper mainly consists, is